MATH 504: CHAPTER 2

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0.1. **Independece results.** The theory we developed so far was developed in ZFC^- (namely $Ax0, Ax_1, Ax3 - Ax9$). In particular, every object we defined was proven to exist in ZFC^- and every theorem we proved wa proven in ZFC^-. By the soundness lemma of first-order logic, this means that in every model of ZFC^-, we can carry all this theory. Some statements, like CH or SH, were not yet settled and we do not know at this point if ZFC^- proves or disproves them. Our goal is to prove that these statements are independent with ZFC^-. Formally, we claim that there is no proof for CH from ZFC and there is no proof for $\neg CH$ from ZFC. To establish this, we would have to present two models: One that satisfies $ZFC + CH$ and one that satisfies $ZFC + \neg CH$. Once we find these models, we can be sure that there is no such proof (again, by the soundness lemma).

Example 0.1. Consider the three axioms of a group Gr in the language $\{e, *\}$:

(1) $\forall x.\forall y.\forall z.x * (y * z) = (x * y) * z$ (Associativity).
(2) $\forall x.e * x = x * e = x$ (Identity element).
(3) $\forall x.\exists y.x * y = y * x = e$ (Inverse element).

Then $\phi := \forall x.\forall y.x * y = y * x$ is independent of Gr, since $S_3 \models \neg\phi$ while $\langle\{0\}, +\rangle \models \phi$.

While producing models of Gr is relatively easy, it is more challenging to produce models of ZFC. In this chapter, we present some of the most basic construction of ZF and ZFC. Since the language of set theory is just $\mathcal{L} = \{\in\}$, a model in the model theoretic sense is a *set* $M \in V$ and an interpretation E for $\in$ as a binary relation of M such that $\langle M, E\rangle \models ZF$ or ZFC. There is no way to construct such a (set)model within ZFC, since this would meen that ZFC proves that there is a model of ZFC which contradicts the second incompleteness theorem (A reach enough theorem cannot prove its own consistency). So there would be two ways to approach this. The first, is to use classes, instead of sets. We will have to formally justify this usage. The second approach is to find set models which satisfy "enough-ZFC" to carry an argument. We will return to the latter later when we will talk about reflection theorems.

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1. Relativization and Class models

Recall that a class M is just a formula ψ with a free variable x, we this of M as the collection $M = M_\psi = \{x \mid \psi(x)\}$. For a formula ϕ in the language of set theory and a class and a class $M = M_\psi$, we define the relativization of ϕ to M, and denote the formula by ϕ^M as follows:

Definition 1.1.
(1) $(x = y)^M$ is $x = y$.
(2) $(x \in y)^M$ is $x \in y$.
(3) $(\alpha \wedge \beta)^M$ is $\alpha^M \wedge \beta^M$.
(4) $(\neg\phi)^M$ is $\neg(\phi^M)$.
(5) $(\exists x.\phi)^M$ is $\exists x.x \in M \wedge \phi^M$

So we simply change in a formula all the $\exists x$ with $\exists x \in M$. So $(\forall x.\phi)^M$ means $\forall x.x \in M \Rightarrow \phi$.

Definition 1.2.
(1) $M \models \phi$ means ϕ^M.
(2) If S is a set of sentences then "$M \models S$" means that for every $\phi \in S$, ϕ^M holds.

The reason we might think of M as a model and the relation $M \models \phi$ in the usual meaning is the following lemma:

Lemma 1.3. *Suppose that S and T are two set of sentences and M is a class (namely a formula). Suppose that T proves that $M \neq 0$ and that $M \models S$ (in the sense of the previous definition), then $Con(T) \Rightarrow Con(S)$.*

Proof. If S is inconsistent that there is a formula (any formula) χ such that S proves that $\chi \wedge \neg\chi$, then we can argue as in the soundness lemma, that T proves that for any formula α that is provable from S, $M \models \alpha$. In particular, $M \models \chi \wedge \neg\chi$ and therefore T proves that $(\chi \wedge \neg\chi)^M$ which is by definition $\chi^M \wedge \neg(\chi^M)$. Thus T proves a contradiction so T is inconsistent. □

So suppose for example that we have constructed a class M from ZFC and proved that $M \models ZFC + CH$, then we proved that $Con(ZFC) \Rightarrow Con(ZFC + CH)$.

Note that we are thinking of the interpretation of $\in$ in M also as $\in$, but we might as well think of another relation E and do all the above with this E. Let us forget about this more complicated situation. Pur first ZF class model (the definition is carried in ZF^-) is:

2. the Well-founded sets and the axiom of foundation

Let us define the V_α hierarchy (in ZF^-):

Definition 2.1. Let $V_0 = \emptyset$, $V_{\alpha+1} = P(V_\alpha)$ and for limit δ, $V_\delta = \cup_{\alpha<\delta} V_\alpha$. Denote the class $WF = \cup_{\alpha \in On} V_\alpha$

Formaly, WF is the formula

$$\exists\alpha \in On.x \in V_\alpha$$

Example 2.2. If $x, y \in V_\alpha$ then $\{x, y\} \in V_{\alpha+1}$ and $\langle x, y\rangle \in V_{\alpha+2}$.
For every α, $|V_\alpha| = \beth_\alpha$.

Lemma 2.3. *(1) For every α, V_α is a transitive set and WF is a transitive class.*
(2) If $X \subseteq WF$ then $X \in WF$.
(3) if $\alpha \leq \beta$ then $V_\alpha \subseteq V_\beta$.
(4) $On \cap V_\alpha = \alpha$.

Proof. (1) By induction on α, $V_0 = \emptyset$ is transitive. At limit stages, we take the union of trasitive sets which is transitive. At successor step, suppose that V_α is transitive, and let us prove that $V_{\alpha+1} = P(V_\alpha)$ is transitive: let $x \in P(V_\alpha)$, and $y \in x$. Then $x \subseteq V_\alpha$, and therefore $y \in V_\alpha$. By the I.H., $y \subseteq V_\alpha$ and therefore $y \in P(V_\alpha) = V_{\alpha+1}$.
(2) Let X be a set such that $X \subseteq WF$, then for every $x \in X$, there is an ordinal α_x such that $x \in V_{\alpha_x}$. Let $\alpha = \sup_{x\in X} \alpha_x$, then $X \subseteq V_\alpha$ and therefore $X \in V_{\alpha+1}$.
(3) Easy, by induction on β.
(4) By induction on α, for $\alpha = 0$ this is clear. At limit stages δ, $V_\delta \cap On = \cup_{\alpha<\delta} V_\alpha \cap On = \cup_{\alpha<\delta}\alpha = \delta$. At successor step $\alpha + 1$, suppose that $V_\alpha \cap On = \alpha$ and let us prove that $V_{\alpha+1} \cap On = \alpha + 1$. Let $\beta \in On$, we have that

$$\beta \in V_{\alpha+1} \Leftrightarrow \beta \subseteq V_\alpha \cap On$$

$$\Leftrightarrow \beta \subseteq \alpha \Leftrightarrow \beta \leq \alpha \Leftrightarrow \beta \in \alpha + 1.$$

For the other direction, if $\beta \in \alpha + 1$, then $\beta \leq \alpha$ and thus β

□

Definition 2.4. For $x \in WF$, $Rank(x) = \beta$ for the minimal β such that $x \in V_{\beta+1}$. (note that the minimal is always successor)

Lemma 2.5. *(1) $V_\alpha = \{x \in WF \mid Rank(x) < \alpha\}$.*
(2) $Rank(y) = \sup(Rank(x) + 1 \mid x \in y)$.

Proof. For (1), if $x \in V_\alpha$ then $x \in V_{Rank(x)+1}$ and by minimality, $Rank(x) < Rank(x) + 1 \leq \alpha$. In the other direction, if $Rank(x) < \alpha$ then $x \in V_{Rank(x)+1} \subseteq V_\alpha$. For (2), Note that if $y \in WF$, then by transitivity, $Rank(x)$ is defined for every $x \in y$. Let $\alpha = Rank(y)$, then $y \in V_{\alpha+1} = P(V_\alpha)$ and thus for every $x \in y$, $x \in V_\alpha$ so $Rank(x)+1 \leq \alpha$. Hence $sup \leq \alpha$. If toward a contradiction $\beta = sup < \alpha$, then for every $x \in y$, $x \in V_\beta$ and therefore $y \subseteq V_\beta$. It follows that $y \in V_{\beta+1}$ contradiction the minimality of $\alpha = Rank(y)$. □

Example 2.6. If $X, Y \in V_\alpha$, then for every $x \in X, y \in Y$, $Rank(x), Rank(y) < \alpha$, hence $Rank(\langle x, y\rangle) \leq \alpha + 1$. So $X \times Y \in V_{\alpha+2}$. Also every $f \in {}^Y X$ is a subset of $X \times Y$, hence $f \in V_{\alpha+2}$. Hence ${}^Y X \subseteq V_{\alpha+2}$, so ${}^Y X \in V_{\alpha+3}$. Also if $X \in V_\alpha$, then for every $x \in X$, $Rank(x) < \alpha$ and therefore every $y \in \cup X$, $Rank(y) < \alpha$ so $\cup X \in V_\alpha$.

Exercise 1. *If* $X, Y \in V_\alpha$*, then:*

- $X \cap Y, X \cup \alpha, X \setminus \alpha, X\Delta Y \in V_\alpha$ *(Since the sup is taken over less elements so* $Rank(X \cap Y), ... \leq Rank(X)$*), and in general, if* $Z \subseteq X$ *then* $Rank(Z) \leq Rank(X)$*.*
- $P(X) \in V_{\alpha+1}$ *(Since* $Rank(P(X)) = \sup(Rank(Y) + 1 \mid Y \subseteq X) = \alpha + 1$*.*
- $X \times Y \in V_{\alpha+2}$*. (Since for every* $x \in X, y \in Y$*,* $Rank(\langle x, y\rangle) \leq \alpha$*, then* $X \times Y \subseteq V_{\alpha+1}$ *and therefore* $X \times Y \in V_{\alpha+2}$*.*
- *If* E *is a relation on* X*, then* $E \in V_{\alpha+2}$ *(Since* $E \subseteq X \times Y$*).*
- *If* E *is an equivalence relation, for* $a \in X$*,* $[a]_E \in V_\alpha$*.*
- *Also* $X/E \in V_{\alpha+1}$*.*

Lemma 2.7. $\mathbb{N} \in V_{\omega+1}$*,* $\mathbb{Z}, \mathbb{Q} \in V_{\omega+4}$ *and* $\mathbb{R} \in V_{\omega+6}$*.*

$$\textbf{Ax2.(Foundation)} \forall x.(x \neq \emptyset \Rightarrow \exists z \in x. \forall y \in x. y \notin z)$$

The axiom of foundation says that every set has a minimal element with respect to $\in$.

Theorem 2.8. *The following are equivalent:*

(1) *Foundation.*
(2) $V = WF$*.*

To prove the theorem we need the following definition:

Definition 2.9. Let A be any set, define the $tr(A) = \cup_{n<\omega} A^{(n)}$, where $A^{(0)} = A$ and $A^{(n+1)} = \cup A^{(n)}$.

Namely $tr(A)$ collects the elements of A and the elements of elements of A and the elements of elements of elements of A, and so on.

Lemma 2.10. $A \subseteq tr(A)$ *is a transitive set (and is the minimal transitive set containing* A*).*

Proof of theorem. (2) $\Rightarrow$ (1), let $x \neq \emptyset$ be any element, then $x \in WF$. Let $\alpha = \min(Rank(y) \mid y \in x)$. Then there is $z \in x$ such that $Rank(z) = \alpha$. To see that $z \cap x = \emptyset$, suppose otherwise, then there is $y \in z \cap x$, and $Rank(y) < Rank(z) = \alpha$, but since $y \in x$, we contradict the minimality of α.

(1) $\Rightarrow$ (2), let x be any set, we need to prove that $x \in WF$. Since $x \subseteq tr(x)$, and since WF is a transitive class, it suffices to prove that $tr(x) \subseteq WF$. Suppose toward a contradiction that this is not the case, then $X = tr(x) \setminus WF \neq \emptyset$. Then by the axiom of foundation, there is $z \in X$ such that for every $X \cap z = \emptyset$. In particular if $y \in z$, then $y \in tr(x)$ but $y \notin X$ and therefore $y \in WF$. Thus $z \subseteq XF$ which implies that $z \in WF$, contradiction. $\square$

2.1. **Absoluteness.** We say the $\phi(x_1, ..., x_n)$ is absolute for $M \subseteq N$ if

$$\forall a_1, ..., a_n \in M.(M \models \phi(a_1, .., a_n) \leftrightarrow N \models \phi(a_1, ..., a_n)$$

Clearly, $x \in y$ and $x = y$ are absolute for every model. Clearly, every statement which is provabley equivalent to an absolute formula is also absolute.

Claim 2.10.1. *If ϕ, ψ are absolute for M, N then so is $\neq \phi$ and $\phi \wedge \psi$. Thus all quantifier free formulas are absolute*

Definition 2.11. A bounded quantifier is a quantifier of the form $\exists x \in y$ of $\forall x \in y$ (was defined in chapter 1).

Lemma 2.12. *If M, N are **transitive** models and all the quantifiers in ϕ are bounded, then ϕ is absolute.*

Proof. It remains to show that if ϕ is absolute then $\exists x \in y.\phi$ and $\forall x \in y.\phi$ are absolute. Note that $\forall x \in y.\phi \equiv \neg(\exists x \in y.\neg\phi)$, now $\neg\phi$ is abosulute and thus it suffices to prove the existential part. Let $y \in M$ be any element. Clearly if $M \models \exists x \in y.\phi$, then since $M \subseteq N$ so does N. Suppose $N \models \exists x \in y.\phi$, then there is $n \in N$ such that $n \in y$ and ϕ^N. Since $y \in M$ and M is transitive, then $n \in M$ and thus ϕ^M holds. Hence $M \models \exists x \in y.\phi$. □

Corollary 2.13. *$x \subseteq y$ and "x is transitive" is absolute for transitive models*

Proof. $x \subseteq y \equiv \forall z \in x.z \in y$ and x is transitive iff $\forall y \in x.\forall z \in y.z \in x$. □

We have defined many set theoretic operations and definition. In each model of ZF^- they can be interpreted and maybe have different meaning. For example $(x = \{y, z\})^M$ then x is the unique set which $M \models$ only $y, z \in x$. but it is possible that a large model N sees that $x = \{y, z, w\}$ for some $w \notin M$, and so the set $\{x, y\}$ in N is different from $\{x, y\}$ in M. In case of $\{x, y\}$, this is not the situation for transitive models, so we need some way to identify when a defined notion is absolute:

Definition 2.14. Let $F(x, x_1, x_2, ..., x_n)$ be a defined notion in models M, N, namely, suppose that $\phi(x, y, x_1, ..., x_n)$ is a formula such that for every $a_1, ..., a_n$, M, N both satisfy that $\forall x\exists!y.\phi(x, y, x_1, ..., x_n)$, then F is says to be absolute if ϕ is.

In particular, for every $a, a_1, ..., a_n \in M$, $F^M(a, a_1, ..., a_n)$ is the unique $y \in M$ such that $\phi^M(a, y, a_1, ..., a_n)$. Now if F is absolute then also $\phi^N(a, y, a_1, .., a_n)$ holds and therefore $F^N(a, a_1, ..., a_n) = F^M(a, a_1, ..., a_n)$.

Proposition 2.15 (Defined notions)**.** *The following are absolute for transitive models.*

(1) $\{x, y\}$, $\{x\}$.
(2) $\langle x, y\rangle$.
(3) $x = \{y, z\}$.
(4) $\emptyset$.
(5) $x \cup y, x \setminus y, x \cap y$

(6) $x+1 := x \cup \{x\}$.
(7) $\cup x, \cap x$.

Proof. $z = \{x, y\}$ iff z is the unique such that $x \in z \wedge y \in z \wedge (\forall t \in z.t = x \vee t = y)$... □

Note that if we compose absolute defined notions we obtain absolute define notions, and if we substitute absolute defined notions in an absolute formula we obtain an absolute formula. So the following are also absolute for trasitive models:

Proposition 2.16. *"z is an ordered pair", $A \times B$, "R is a relation", "R is an order", "R is a linear order", "R is a function", "R(x)", "R is 1 − 1/onto/bijection".*

For example $P(X)$ is not abolute since it is defined by the formula $Y = P(X)$ iff $\forall Z.Z \subseteq X iff X \in Y$ and we do not have way to bound the quatifier $\forall Z$. Similarily, ${}^Y X$ is not absolute. However, since $Z \subseteq X$" and $f : X \to Y$ are transitive, we have that:

Lemma 2.17. *If M is transitive then:*
(1) $P^M(X) = P(X) \cap M$.
(2) $({}^X Y)^M = {}^X Y \cap M$.

Now let us consider absoluteness with respect to V.

Lemma 2.18. *Suppose that M is a transitive model of $ZF^- - P - Inf$, then the following are absolute for M:*
(1) *"R well orders A".*
(2)

Proof. Suppose that R well orders A We need to check that

$$M \models \forall X.X \subseteq A \wedge X \neq \emptyset \Rightarrow \exists x \in X.\forall y \in X.y = x \vee (\langle x, y\rangle \in R)$$

Th inner formula is absolute for transitive model and the only problem is $\forall X$ which is not bounded. But note that $\forall X \in M$ follows from $\forall X$ which we are assuming (namely, $\forall$ is downward absolute). □

Corollary 2.19. *Let M be a transitive model of ZF^-, then "x is an ordinal", "x is a limit/successor ordinal", "x is a finite ordinal", ω are absolute between V and M.*

Also "α is a cardinal" is not absolute since this means that for every $\beta < \alpha$ and every function $f : \beta \to \alpha$. f is not onto. You might think that we can bound $f \in {}^\alpha\beta$, but the defined notion ${}^\alpha\beta$ is not absolute for transitive models. However, if $M \subseteq N$, then if $N \models \alpha$ is a cardinal then $M \models \alpha$ is a cardinal. So for example ω_1 is not an absolute notion.

Corollary 2.20. *Let M be a transitive model of $ZF^- - Inf$. If then M satisfy the the axiom of infinity iff $\omega \in M$*

Next we would like to verify that WF is a model of ZF or ZFC. For this we will formulate sufficient condition for a general class models. The next lemma simplify the verification that a class M satisfies certain axioms:

Theorem 2.21 (ZF^-). *(1) $M \models Ax0$ iff $M \neq 0$.*

(2) If M is a transitive class, then $M \models$extensionality.

(3) If $\forall x, y \in M \exists z \in M.\{x, y\} \subseteq z$ then $M \models$pairing.

(4) If $\forall x \in M \exists y \in M. \cup x \subseteq y$, then $M \models$ union.

(5) If $M \subseteq WF$, then $M \models$foundation.

(6) If M is transitive, and $\forall x \in M.\exists y \in M.P(x) \cap M \subseteq y$, then $M \models$powerset.

(7) If for every formula $\phi(x, x_1, .., x_n)$ $\forall a_1, .., a_n \in M.\forall m \in M.\{x \in m \mid \phi^M(x, a_1, ..., a_n)\} \in M$, then $M \models$ comprehension.

(8) If for every formula $\phi(x, y, x_1, .., x_n)$ $\forall a_1, .., a_n \in M.\forall m \in M$ such that $.M \models \forall a \in m \exists! z.\phi(x, y, a_1, ..., a_n)$, there is $Y \in M$ such that $\{y \mid y \in m \wedge \phi^M(x, y, a_1, ..., a_n)\} \subseteq Y$, then $M \models$ replacement.

Proof. (1) $M \models Ax0$ iff $(\exists x.x = x)^M \equiv \exists x \in M.x = x$ iff $M \neq \emptyset$.

(2) $M \models Ax_1$ iff $(\forall x, y.(\forall z.z \in x \Leftrightarrow z \in y) \Rightarrow x = y)^M \equiv \forall x, y \in M.(\forall z \in M.z \in x \Leftrightarrow z \in Y) \Rightarrow x = y$. Suppose M is transitive [1], Let $x, y \in M$, suppose that $\forall z \in M.z \in x \Leftrightarrow z \in y$. Since M is transitive, then for every $z \in x$ of $z \in Y$, then $z \in M$ and therefore $\forall z.z \in x \Leftrightarrow z \in y$. By extensionality in V we get that $x = y$.

(3) Exercise.

(4) Exercise.

(5) Suppose that $M \subseteq WF$, then $M \models$foundation iff $(\forall x.x \neq \emptyset \exists y \in x.x \cap y = \emptyset)^M$ Let $x \in M$ and suppose that $(x \neq \emptyset)^M$, then $\exists y \in x \cap M$. We want to prove that $(\exists y \in x.\forall z \in x.z \notin y)^M$. Since $M \subseteq WF$, there is $y \in x \cap M$ of minimal rank and as in the Theorem 2.8, we have that for every $z \in x \cap M$, $z \in WF$ and therefore $z \notin y$.

(6) Since M is transitive $(z \subseteq x)^M$ is just $z \subseteq x$ and therefore the assumption ensures that $M \models$powerset axiom.

(7) We want to prove that for every $z \in M$ there $y \in M$ such that

$$(\forall x.x \in y \leftrightarrow x \in z \wedge \phi(x, a_1, ..., a_n))^M$$

then the assumption of (6) ensures that there is such $y \in M$.

□

Theorem 2.22. *If V is a model of ZF^- (ZFC^-) then WF is a model of ZF (ZFC).*

Proof. $Ax0 -$ $\emptyset \in WF$.

$Ax1 -$ WF is transitive

$Ax2 -$ $WF \subseteq WF$

$Ax3-$ Let $\phi(x, x_1, .., x_n)$ be a formula and $a_1, ..., a_n \in WF$ parameters. Let

[1]If M is not transitive then M might "miss" some elements $z \in x$ such that $z \notin y$ and thus fail to satisfy extensionality.

$m \in WF$, we need to prove that $\{x \in m \mid \phi^{WF}(x, a_1, ..., a_n)\} \in WF$. Note that this set is a subset of WF and therefore is belongs to WF.
$Ax4-$ we already checked.
$Ax5-$ We already checked.
$Ax6-$ Similar to comprehension, suppose that $\phi(x, y, x_1, ..., x_n)$ is a formula such that for every $a_1, ..., a_n \in WF$, every $m \in WF$:

$$\forall x \in m.\exists! y \in W_F.\phi^{WF}(x, y, a_1, ..., a_n)$$

Then in V we can write the formula

$$\psi(x, y, m, a_1, ..., a_n) \equiv y \in WF \wedge \phi^{WF}(x, y, m, a_1, ..., a_n)$$

Then in V we have that for every $x \in m$ $\exists! y.\psi(x, y, a_1, ..., a_n)$, o by replacement in V there the set $Y = \{y \mid \exists x \in m.\psi(x, y, a_1, ..., a_m)\}$ exists. By the definition of ψ, we will have that $y \in WF$ and therefore $Y \subseteq WF$ and so $Y \in WF$.

$Y \subseteq WF$ which implies that $Y \in WF$. Y is as wanted.
$Ax7-$ we have that $\omega \in V_{\omega+1}$ and since WF is transitive, then $WF \models$ infinity
$Ax8-$ Let $x \in WF$, then $P(x) \cap WF = P(x)$ and $P(x) \in WF$. So there is $y \in WF$ such that $P(x) \cap WF \subseteq y$.
$Ax9-$ Let us check that every set can be well ordered. Let $A \in WF$, then in V there is a well ordering R of A. Then $R \in WF$ (since $R \subseteq A \times A$). We already seen that "R well-orders A" is absolute. It follows that $WF \models A$ can be well ordered.

□

Theorem 2.23 (ZFC)**.** *If κ is strongly inaccessible then $V_\kappa \models ZFC$*

Corollary 2.24. *$ZFC \not\vdash$ "$\exists$ inaccessible cardinal"*

Proof. Otherwise $ZFC \vdash Con(ZFC)$. □

Actually more is true, we cannot construct a model M such that $M \models$ there is an inaccessible cardinal.

2.2. **Well founded relations and reflection theorems.** Induction can be preformed on non linear sets as well, and here is the most general framework:

Definition 2.25. A relation R on a set A is called well founded if:

$$\forall X \subseteq A.X \neq \emptyset \Rightarrow \exists y \in X.\forall z \in X.\neg zRy$$

a totally ordered set which is well founded is an ordered set.

Definition 2.26. R is called *set-like* in A if $\{x \in A \mid xRy\}$ is always a set for every $y \in A$.

Corollary 2.27. *$\in$ is well-founded on any set (class) iff $V = WF$.*

Definition 2.28 (R-induction/recursion)**.** Given a well founded and set like relation R on A, . Induction: $(\forall y.(\forall xRy.\Psi(x)) \Rightarrow \Psi(y)) \Rightarrow \forall y.\Psi(y)$
Recursion: Suppose that $g(x)$ is defined for every xRy. Define $g(y)$, given the knowledge of $\{g(x) \mid xRy\}$.

Definition 2.29 (Mostowski's collapse)**.** Let A be any class and R a set-like and well-founded relation. Define $\pi(x) = \{\pi(y) \mid yRx\}$.

Theorem 2.30. *$M = \{\pi(x) \mid x \in A\}$ is a transitive class and $\pi :< A, R > \to < M, \in>$ is an homomorphism.*

Example 2.31. If $R = 0$ then $\pi(x) = 0$ for every $x \in A$. If R well orderes A then π is the isomorphism to $otp(A, R)$.

Definition 2.32. We R is extensional on A if

$$\forall x, y \in A.x = y \leftrightarrow (\forall z.(zRx \Leftrightarrow zRy)$$

Example 2.33. If A is transitive, then $\langle A, \in \rangle$ is existensial.

Theorem 2.34 (The Mostowski collapse)**.** *If R is a well-founded set-like extensional relation on A then there is a transitive class M such that the Mostowski collapse π is an isomorphism between A and M. Moreover, if $X \subseteq A$ is transitive then $\pi \restriction X = id$.*

2.3. **Reflection theorems.** This is the second type of "models" we will have for ZFC, once which only satisfy a fragment of ZFC.

A subformula of a formula ϕ is just a substring which is a legitimate formula.

Definition 2.35. A list of formulas is subformula closed, if every subformula of a a formula in the list appears in the list.

Lemma 2.36. *Suppose that $M \subseteq N$ be classes and $\phi_1, ..., \phi_n$ be a subformula closed list of formulas. The following are equivalent:*

(1) $\phi_1, ..., \phi_n$ are absolute for M, N
(2) Whenever ϕ_i is of the form $\exists x\phi_j(x, x_1, .., x_m)$, then:

$$\forall y_1, ..., y_m \in M.(\exists x \in N.\phi^N(x, y_1, ..., y_m) \Rightarrow \exists x \in M.\phi^N(x, y_1, ..., y_m)).$$

Proof. (1) $\Rightarrow$ (2), we now absoluteness, in particular $\phi_j^N(x, y_1, .., y_n)$ iff $\phi_j^M(x, y_1, ..., y_n)$ so this is clear. (2) $\Rightarrow$ (1) is like tarski's criterion for elementary submodel, we have to go by induction on subformulas, for atomic formulas this is trivial. In the inductive step, $\neq$ and $\wedge$ are clear. Now (b) gives you the missig part to complete the induction. ☐

Theorem 2.37 (The reflection theorem for V_α)**.** *Let $\phi_1, .., \phi_n$ be any formulas, then*

$$\forall \alpha \exists \beta > \alpha.\phi_1, ..., \phi_n \text{ are absolute for } V_\beta$$

Theorem 2.38 (The General reflection theorem)**.** *Suppose that $Z = \cup_{\alpha \in On} Z_\alpha$ such that:*

(1) $\alpha \leq \beta \Rightarrow Z_\alpha \subseteq Z_\beta$.
(2) Z_α *is a set.*
(3) *For limit* δ, $Z_\delta = \cup_{\alpha<\delta} Z_\alpha$

Then for any formulas $\phi_1, .., \phi_n$

$$\forall\alpha\exists\beta > \alpha.\phi_1, ..., \phi_n \text{ are absolute for } Z, Z_\beta$$

Proof. We just need to find β which is a closure point for the Skolham functions. □

Corollary 2.39. *Same for statements in ZFC*

3. Gödel's constructible universe

This is due to Godel.

Definition 3.1. Let A be a set. A subset $B \subseteq A$ is said to be definable from A is there if a formula $\phi(x, x_1, ..., x_n)$ and $a_1, ..., a_n \in A$ such that $B = \{a \in A \mid \phi^A(a, a_1, ..., a_n)\}$.

Let us define an opperation $\mathcal{D}(A) = \{B \in P(A) \mid B \text{ is definable in } A\}$,

Definition 3.2. The constructable universe is defined as follows:

(1) $L_0 = \emptyset$.
(2) $L_{\alpha+1} = \mathcal{D}(L_\alpha)$.
(3) For a limit δ, $L_\delta = \cup_{\alpha<\delta} L_\alpha$.

Let $L = \cup_{\alpha<\delta} L_\alpha$, then L is called "the constructible universe".

We would like to define a formula $\phi(A, B, a_1, ..., a_n)$ which said "B is definable in A from $a_1, ..., a_n$" and this formula is absolute for transitive models. We could have taken the approach of Godel numbers, but let us do it directly:

Definition 3.3. Let:

(1) $Diag_\in(A, n, i, j) = \{s \in A^n \mid s(i) \in s(j)$.
(2) $Diag_=(A, n, i, j) = \{s \in A^n \mid s(i) = s(j)\}$.
(3) $Proj(A, R, n) = \{s \in A^n \mid \exists r \in R.r \restriction n = s\}$

Note that These are all absolute defined notions.

Definition 3.4. Define $D'(k, A, n)$ recursively on k (for all n):

(1) $D'(0, A, n) = \{Diag_\in(A, n, i, j), Diag_=(A, n, i, j) \mid i, j < n\}$.
(2) $D'(k+1, A, n) = D'(k, A, n) \cup \{A^n \setminus R \mid R \in D'(k, A, n)\} \cup \{R \cap S \mid R, S \in D'(k, A, n)\} \cup \{Proj(A, R, n) \mid R \in D'(k, A, n+1)\}$

Again, note that $D'(k, A, n)$ is absolute.

Definition 3.5. $Df(A, n) := \cup_{k<\omega} D'(k, A, n)$. And $Df(A, n)$ is absolute

The idea of $Df(A, n)$ is that if R is an n-arry relation defined from A without parameters, then $R \in Df(A, n)$. By the usual way formulas are defined, the set $Df(A, n)$ is exactly those R's. We have that $|Df(A, n)| \leq \omega$.

Definition 3.6. Let $\mathcal{D}(A) := \{X \in P(A) \mid \exists n < \omega.\exists R \in D(A, n+1)\exists s \in A^n.X = \{x \in A \mid s^\frown x \in R\}\}$.

Again, note that $\mathcal{D}(A)$ is absolute.

Corollary 3.7. *Let M be a transitive model, then $\mathcal{D}(X)$ is an absolute defined notion for M.*

Corollary 3.8. *Let M be a transitive model of ZF then $\cup_{\alpha\in M\cap On} L_\alpha \subseteq M$ and if $On \subseteq M$ then $L \subseteq M$.*

Proposition 3.9.
(1) $D(A) \subseteq P(A)$.
(2) If A is transitive then $A \subseteq D(A)$.
(3) If $X \subseteq A$ and X is finite then $X \in D(A)$.
(4) If $|A| \geq \omega$ then $|A| = |D(A)|$.

Proof. (1) is clear. For (2), consider the formula $\phi(x, y)$ to be $x \in y$. Then for every $a \in A$, a is definable from a $\{x \in A \mid (x \in a)^A\} = \{x \in A \mid x \in a\} = a$ (since A is transitive). $(3), (4)$ are a bit trickier than one thinks, let us leave the formal proof for the curious reader and refer to Kunen. The intuition behind (4) is that we can enumerate all the formulas, and there are countably many. Now consider $A^{<\omega}(Formulas)$, then the left side (which provides the parameters) and the righthand side the formula gives all the elements of $D(A)$, but this set has size $|A|$, so $|D(A)| \leq |A|$. □

Proposition 3.10 (Properties of L_α).
(1) L_α is transitive, and L is a transitive class.
(2) $\alpha \leq \beta$ then $L_\alpha \subseteq L_\beta$. $L_\alpha \in L_{\alpha+1}$. Every finite subset of L_α is in $L_{\alpha+1}$.
(3) For every $n < \omega$, $L_n = V_n$ and thus $L_\omega = V_\omega$.
(4) For every α, $V_\alpha \subseteq L_\alpha$.

Definition 3.11. For $x \in L$, define $Rank_L(x) = \alpha$ for the minimal α such that $x \in L_{\alpha+1}$.

Proposition 3.12.
(1) $L_\alpha = \{x \in L \mid Rank_L(x) < \alpha\}$.
(2) $On \cap L_\alpha = \alpha$, hence $On \subseteq L$ and $Rank_L(\alpha) = \alpha$.

Theorem 3.13. *For every $\alpha \geq \omega$, $|L_\alpha| = |\alpha|$.*

Proof. Clearly, for every α, $\alpha \subseteq L_\alpha$ thus $|\alpha| \leq |L_\alpha|$. We prove the other direction by induction on α, for $\alpha = \omega$ this is easy since $L_n = V_n$ are finite and thus $V_\omega = \cup_{n<\omega} V_n$ is countable. For limit $\alpha > \omega$, this follows by induction hypothesis and for successor α, we have that $|L_{\alpha+1}| = |D(L_\alpha)| = |L_\alpha| = |\alpha| = |\alpha + 1|$. □

Corollary 3.14. $V_{\omega+1} \neq L_{\omega+1}$

Proof. Just by cardinality consideration □

3.1. **ZFC in L.** Here we start with ZF and we establish that $L \models ZFC$, thus we obtain:

Theorem 3.15. $Con(ZF) \Rightarrow Con(ZFC)$.

Let us start only with ZF

Theorem 3.16. $L \models ZF$

Proof. $Ax0$- $L \neq 0$.
$Ax1$- L is transitive.
Ax_2- $L \subseteq WF$.
Ax_3- Let $\phi(x, x_1, ..., x_n)$ be a formula, and Then in V we have that set $\{x \in L \mid \phi$
Ax$_4$- If $x, y \in L_\alpha$, then $\{x, y\}$ is definable from x, y so $\{x, y\} \in L_{\alpha+1}$.
$Ax5$- Let $x \in L$ then $\cup x \subseteq L$, take $\alpha = \sup(Rank_L(y) \mid y \in \cup x)$. Then $\cup x \subseteq L_\alpha$.
$Ax6-$ Let $\phi(x, y, a_1, .., a_n)$ be a formula suitable for reflection in L then for every $x \in a \in L$ $\exists! y \in L$. $\phi^L(x, y, a_1, ..., a_n)$ and therefore in V we can form

$$\{y \in L \mid \exists x \in a. \phi^L(x, y, a_1, ..., a_n)\}$$

Then we can find again α large enough such that this set is included in L_α.
$Ax7$- $\omega \in L$.
$Ax8$- We just have to find α such that $P(X) \cap L \subseteq L_\alpha$. □